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(3 X 3) Rank Two Tensor Diagonalization of a Real Symmetric Tait-Bryan Angles Required for NASA Technical Memorandum 10389
any permutation of 3-1-3 rotation angles.

The Tail-Bryan angles differ from the more familiar Euler angles which are rotation angles used to define the pitch, flap, and lead-lag angles in helicopter rotor quadratric coefficients. Tail-Bryan angles, which are any permutation of 1-2-3 encountered tensors such as inertia tensors, rank two $(3 \times 3)$, and symmetric tensors, and therefore is applicable to commonly encountered rank two tensors. This process is required to diagonalize a real symmetric

Real Symmetric $(3 \times 3)$ Rank Two Tensor Required for Diagonalization of a Tail-Bryan Angles

ABSTRACT
SUMMARY

2.4 Determination of Euler Angles Which Will Diagonalize a Real (3 x 3) Rank Two Tensor

2.3 Basic Properties of the Characteristic Equation of a Real Symmetric Matrix and Rotation Sequences

2.2 Determination of Rotation Angles from a Specified Transformation

2.1 Basic Concepts for Co-ordinate Rotations and Transformations

TECHNICAL

INTRODUCTION

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ABSTRACT

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is written as $A^k$.

$A^k$ denotes the $k$-th column partition of $A$ (the corresponding column vector). $A^T$ denotes the transpose of the corresponding transformation matrix. $A^{-1}$ denotes the inverse (the corresponding transformation matrix is written $A^{-1}$).

In coordinate system $\mathcal{I}$, to coordinate system $\mathcal{J}$, the corresponding transformation matrix $A$ is written as $A^{\mathcal{I}\mathcal{J}}$. $A^{\mathcal{I}\mathcal{J}}$ is a matrix which denotes that the transformation is from co-ordinates of system $\mathcal{I}$ to co-ordinates of system $\mathcal{J}$.

Post-superscript $k!$ denotes an exponent (the corresponding transformation matrix is written $A^{k!}$).

Rotation could be written as $A_k^{\mathcal{I}\mathcal{J}}$. Rotation sequence $\mathcal{I} \rightarrow \mathcal{J} \rightarrow \mathcal{I}$, the transformation matrix for an elemental axes of rotation in sequence order from right to left for $\ldots k! \mathcal{J} \ldots$.

Pre-superscript $k!$ denotes a linear orthonormal transformation (matrix) from one co-ordinate system to another.
| \begin{array}{c}
\text{Example} \\
\text{Note:} A_{ij} \equiv A_{ji}
\end{array} |  \\
| \begin{array}{c}
\text{Note:} A_{ij} \equiv A_{ji}
\end{array} |
rotation index

Eigenvalue solution number

co-ordinate system number denoting co-ordinate system h

rotation index
element position index

Imaginary number base = \sqrt{-1}

rotation index
element position index

optical inclination

c \& H \& \tau

\kappa \invariant of the characteristic equation of a real symmetric (3x3) rank two
general function

(i,j) \text{ element of } F

general rank two tensor

general rank two tensor

(3x3) identity matrix

(i,j) \text{ element of } F
third rotation angle
vehicle yaw angle
blade flare angle
Kramechers Symbol Index
second rotation angle
vehicle pitch angle
column vector partition index
Kramechers Symbol Index
first rotation angle
general 3-vector expressed in co-ordinate system H
V
general 3-vector
[TRM]
Transformation Matrix
[RS]
Rotation Sequence
[RA]
Rotation Angles
post superscript of f denoting "pseudo-inverse"
integer

X
The characteristic equation of a real symmetric (3x3) rank two tension
linearized angular velocity composed of combinations of $\Omega$ in the solution of
the equivalent single rotation angle as specified by Euler's Theorem

general rotation angle
vehicle roll or bank angle
precessional alignment
$\lambda$ eigenvalue
value of repeated eigenvalues
$\phi$ general eigenvector
vector of eigenvalues (note, not the same as an eigenvector)
$\delta$ blade pitch angle
general rotation angle
blade lead-lag angle

Kronecker's symbol, $\delta_{a\neq b} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases}$
Determining rotation angles from specified transformation matrices and rotation sequences is a complex problem. The process involves analyzing matrices and applying specific transformations. The basic coordinate system rotations are defined in Section 2.1, and its subsections discuss the associated transformation matrices.

2.0 Technical Background

Matrix representations of the invariant characteristic of a particular coordinate system. Above mentioned procedure is to determine the Euler-Bryan angles required to diagonalize the characteristic matrix with respect to the rotation of the coordinate axes under consideration and no change in orientation. The characteristic is expressed in the initial or initial-coordinate system. Here, it is assumed that the invariance elements of an invariant characteristic of an entity in a new coordinate system in terms of a new orientation. Their associated transformation matrices provide a simple means to express the sequences of rotations of a Euclidean three-space coordinate system from an initial orientation.

1.0 Introduction

Tait-Bryan angles, Euler angles, and other rotation sequences used herein are specified...
Let $A$ denote the transformation matrix. Then

$$A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}$$

A summary of important properties of these transformations follows.

**3. Basic Concepts for Coordinate Rotations and Transformations**

2.4. Presented in Section 2.4, the tensor $A$ defined in Section 2.3. Finally, the topics discussed here are defined in Section 2.3. Finally, the tensors which are relevant to the topics discussed here are defined in Section 2.3. Finally, the topics discussed in Section 2.2 and its sub-sections. Basic properties of a real symmetric $(3 \times 3)$ tensor?
Equations (5) and (6) expressed in matrix form are

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \times A_x \times 6 \times 6 \\ A \times A_y \times 6 \times 6 \end{bmatrix} = 6 \times 6 \]

where \( \delta_{AE} \) is the summation convention is assumed.

Properties 2 - Orthogonality

\[ V_2 = V_2 \]

then

Property 1 - Basic Vector Transformation

and \( A \times A_x \) is the transformation matrix from coordinate system \( I \) to coordinate system \( 2 \).
(8) \[ \text{Property 4 - Determinant of } A \]

\[ \det(A) = 1 \]

(7) \[ \begin{cases} A^{-1} = A^{-1} \\ A = A^{-1} \end{cases} \]

\[ \text{Property 3 - Transpose Equals Inverse} \]

Equation (6) implies the following very important property:

\[ A^T = A \quad A^T = E_3 \]

where \( E_3 \) is the (3x3) identity matrix.
With \( \lambda = 1 \) as specified by Euler's Theorem, the angle \( \Phi \) is the equivalent single rotation angle about the eigenvalue associated and the angle \( \Phi \) is the eigenvalue single rotation angle about the eigenvalue associated.

\[
(\text{Characteristic Equation}) \quad \lambda^2 + [1-T(A)] \lambda - I = 0
\]

\[
(\text{Characteristic Equation}) \quad \lambda^3 - [T(A)]^2 \lambda^2 + [T(A)] \lambda - I = 0
\]

or equivalently

\[
\lambda^3 - [T(A)]^2 \lambda^2 + [T(A)] \lambda - I = 0
\]

where the characteristic equation is

\[
\begin{bmatrix}
\phi - \lambda & \psi \\
-\psi & \phi + \lambda
\end{bmatrix}
= \begin{bmatrix}
\lambda^3 \\
\lambda^2 \\
\lambda \\
1
\end{bmatrix}
= \mathbf{V}
\]

for \( \lambda \) yields

\[
\text{Det}(A - \lambda I) = 0 \quad (\text{The Eigenvalue Problem})
\]

Property \( 5 \) - Eigenvalues of \( A \)
\[
\begin{align*}
(14) \quad & N = N(N-1) \cdots 3 \cdot 2 \cdot 1, \\
\text{then} \quad & A_{X_k} = A_{(N-1)(N-2) \cdots 3 \cdot 2 \cdot 1}.
\end{align*}
\]

and \( A_{X_k} \) is the transformation matrix from \( \text{co-ordinate system K to co-ordinate system} \) \( N \).

\[N, \ldots, 2, 1, X_k.\]

Property 7 - Successive Rotations

Where the eigenvector \( U \) is the eigenvector associated with \( \lambda = 1. \)

\[
(13) \quad \frac{1}{2} \cos \left( \frac{1}{2} \theta \right) = \phi
\]

by a unique eigenvector \( U \) and an equivalent single rotation angle \( \phi \). Any sequence of coordinate transformations can be characterized using Euler's Theorem.

Property 6 - Euler's Theorem and the Equivalent Single Rotation Angle

(Scalar solutions to Equation 6)

It should be noted that \( \lambda \) is not, in general, an eigenvector. If \( \lambda \) is a vector of eigenvectors.
If \( P \) is a rank two tensor expressed in co-ordinate system \( k', x_1, x_2 \), then the fundamental

\[
V' = [A_{21}] V_2
\]

Noting the property defined by Equation (7), Equation (17) can be expressed as

\[
V' = [A_{21}]^{-1} V_2
\]

The inverse transformation of the transformation defined by Equation (2) is

\[
V_n = A_{11} V_1
\]

which means that Equation (14) can be written as

\[
A_{11} A_{21} \ldots A_{N1} A_{1(1-N)1(1-N)(2)} \ldots A_{N1} A_{1(1-N)} = V_n
\]

and the single equivalent transformation matrix is

\[
V_{\text{eq}}
\]
Diagonal. Premultiplication of Equation (21) by $[\mathbf{A}_{21}]^T$ and rearranging yields

\[ \Phi = \mathbf{A}_{21}^T \mathbf{F}_2 \mathbf{A}_{21} \]

The rank two tensor diagonalization problem requires the determination of the transformation matrix

Property 9 - Rank Two Tensor Diagonalization Problem

(21)

Rank Two Tensor Transformation Law

From which

\[ \Phi = \mathbf{A}_{21}^T \mathbf{F}_2 \mathbf{A}_{21} \]

(22)

Noting Equation (18), Equation (19) can be written as

\[ \mathbf{F}_2 = \mathbf{A}_{12} \mathbf{A}_{21}^T \]

(19)

Fundamental Tensor Invariance Identity

Tensor Invariance Identity requires that

\[ \mathbf{A}_{12} \mathbf{A}_{21} = \mathbf{A}_{21} \mathbf{A}_{12} \]
The second term of Equation (22) can be rearranged according to:

\[
A(\alpha) = \begin{bmatrix} A_{a1} \\ A_{a2} \\ A_{a3} \end{bmatrix}, \quad \alpha = 1, 2, 3
\]

where \( A(\alpha) \), \( \alpha = 1, 2, 3 \), are column vector partitions of \( [A_{21}]^T \) such that

\[
[F'']^T [A_{21}]^T = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}\begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{31} \\ A_{32} \\ A_{33} \end{bmatrix}
\]

The first term in Equation (22) can be written in terms of its elements and then the \( [A_{21}]^T \) matrix...
An important implication of Equations (28) and (29) is that the diagonalization of a rank two

\[
\text{Det}(F(t, x^E)) = 0 \quad \text{Characteristic Equation of } F
\]

(see Section 2.3)

The eigenvalues \( \lambda \) of \( F \) are determined by solving the characteristic equation of \( F \).

\[
F(t, x^E) \lambda = 0 \quad \text{for } \lambda = 1, 2, 3 \quad \text{or not summed}
\]

\[
F(t, x^E) - x^E A_\lambda = 0 \quad \text{for } \lambda = 1, 2, 3 \quad \text{or not summed}
\]

which is equivalent to three equations of the form

\[
0 = \begin{bmatrix}
  A_{(3)} & A_{(2)} & A_{(1)} \\
  x_3 A_{(3)} & x_2 A_{(2)} & x_1 A_{(1)} \\
  x_3 A_{(2)} & x_2 A_{(1)} & x_1 A_{(3)}
\end{bmatrix}
\]
original co-ordinate system about one of its axes. The right-handed co-ordinate system is assumed for the initial co-ordinate system to that specified for a co-ordinate system defined by referring an elemental transformation is a linear orthogonal transformation of co-ordinates from that.

Assuming a Euclidean three-space co-ordinate system whose axes are numbered 1, 2, and 3.


Subsection 2.7.2.

The remainder of this section is divided into the subsections. The definition of the element matrix from a specified rotation sequence and corresponding rotation angles is discussed on the assumed rotation sequence used to define the diagonalising transformation matrix of the co-ordinate system. F is expressed (in) and that this result is not at all dependent on the eigenvalues of F (note, the superscript F can be dropped because the result is independent of tensor as an eigenvalue problem with the diagonalised form being
For a rotation \( \phi \) about the second axis, \( \gamma \) axis

\[
\begin{bmatrix}
\cos \phi & 0 & \sin \phi \\
0 & 1 & 0 \\
-\sin \phi & 0 & \cos \phi
\end{bmatrix} \equiv \gamma_2 \gamma \gamma_2(\phi)
\]

\( P_2 = \gamma_2(a) P \gamma_2(a) \)

and

\( v = \gamma_2(a) v \gamma_2(a) \)

\[
\begin{bmatrix}
0 -5\alpha & \cos \alpha \\
0 & \cos \alpha & 5\alpha \\
0 & 0 & 1
\end{bmatrix} \equiv \gamma_1 \gamma \gamma_1(\alpha)
\]

For a rotation \( \alpha \) about the first axis, \( \gamma \) axis

\( P_1 = \gamma_1(a) P \gamma_1(a) \)

\( v = \gamma_1(a) v \gamma_1(a) \)

Material presented here is a single rotation about one of the co-ordinate axes, the transformation matrices become initial co-ordinate system, \( P \), according to Equation (2.1). For these elemental cases whose for the tensor \( F \) expressed in the rotated co-ordinate system, \( P' \), is related to that expressed in the initial co-ordinate system, \( F' \), according to Equation (2.1). A vector \( \mathbf{v} \) expressed in the rotated co-ordinate system, \( v' \), is related to that expressed in the initial co-ordinate system, \( v \). Herein, A vector.
Initial co-ordinate system through the rotation angle θ about the x axis.

\( A(\theta) \) indicates that the transformation is an elemental transformation obtained by rotating the

rotation.

\( A_2 \) is the transformation matrix from the initial co-ordinate system to that obtained by the element

where

\[ A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \]

For a rotation θ about the third axes, axes

\[ P_2 = \begin{bmatrix} P \end{bmatrix} A_2(\theta) \]

and

\[ \begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} V \end{bmatrix} A_2(\theta) \]

then

\[ P_2 = \begin{bmatrix} P \end{bmatrix} A_2(\theta) \]

and

\[ \begin{bmatrix} V \end{bmatrix} = \begin{bmatrix} V \end{bmatrix} A_2(\theta) \]

then
The six possible combinations of two different elemental transformations are presented below.

Equation (15),

The six possible combinations of two different elemental transformations are presented below.
\[
2^3A = A(A^2)A(x) = \\
\begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix} \\
1^2A = A(A^2)A(x) = \\
\begin{bmatrix}
0 & \cos \theta & \sin \theta \\
-\sin \theta & 1 & \cos \theta \\
\cos \theta & -\sin \theta & 0
\end{bmatrix} \\
1^2A = A(A^2)A(x) = \\
\begin{bmatrix}
0 & \cos \theta & 0 \\
-\sin \theta & 1 & 0 \\
\cos \theta & -\sin \theta & 1
\end{bmatrix} \\
3^2A = A(A^2)A(x) = \\
\begin{bmatrix}
\cos \theta & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \cos \theta
\end{bmatrix}
\]
There are six Euler angles associated with an Euler rotation sequence. These are six axes. Euler angles are the rotation angles about the same numbered axes and the second rotation is about one of the other two rotations. Euler rotation sequences are composed of three elemental rotations where the first and last rotations are about the same numbered axes.

2.1.2.1 Euler angles

The transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence, the transformation matrix and the rotation sequence.

Note also that each \( \mathbf{A} \) has a unique position \( f(k) \) whose element is identically zero. This last 3 elements for \( k = 1, 2, 3 \) but \( k \neq 1 \).

Note that, in general, \( \mathbf{A} \) and \( \mathbf{A}^{-1} \) for \( k = 1, 2, 3 \) are different.

Denote \( \mathbf{A}(k) \) by \( \mathbf{A}(k) \).

\[
\mathbf{A} = \begin{bmatrix}
0 & -\sin \alpha & \cos \alpha \\
\sin \alpha & 0 & -\cos \alpha \\
-\cos \alpha & \sin \alpha & 0
\end{bmatrix}
\]

\[
\mathbf{A}(k) = \begin{bmatrix}
0 & -\sin \alpha & \cos \alpha \\
\sin \alpha & 0 & -\cos \alpha \\
-\cos \alpha & \sin \alpha & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\sin \alpha & -\cos \alpha & 0
\end{bmatrix}
\]

\[
\mathbf{A}(k) = \begin{bmatrix}
0 & -\sin \alpha & \cos \alpha \\
\sin \alpha & 0 & -\cos \alpha \\
-\cos \alpha & \sin \alpha & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\sin \alpha & -\cos \alpha & 0
\end{bmatrix}
\]

\[
\mathbf{A}(k) = \begin{bmatrix}
0 & -\sin \alpha & \cos \alpha \\
\sin \alpha & 0 & -\cos \alpha \\
-\cos \alpha & \sin \alpha & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
-\sin \alpha & -\cos \alpha & 0
\end{bmatrix}
\]
and the associated Euler transformation is:

The angles of the ascending node \( \Omega \), the argument of periastron \( \omega \), and the periapsis argument longitude of the ascending node \( \Omega \) are the orbital inclination \( i \), and \( \Omega \) the periastron argument \( \omega \). In this case, the angular elements are:

A well-known example of an Euler rotation sequence and associated Euler angles is the any

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \theta )</th>
<th>( \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>( \nu )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>( \lambda )</td>
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<td>( \gamma )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>( \nu )</td>
<td>( \gamma )</td>
</tr>
</tbody>
</table>

\begin{align*}
(1) V_0 & \equiv V_{323} \\
(2) V_1 & \equiv V_{312} \\
(3) V_2 & \equiv V_{232} \\
(4) V_3 & \equiv V_{213} \\
(5) V_1 & \equiv V_{131} \\
(6) V_2 & \equiv V_{121} \\
(7) V_3 & \equiv V_{111}
\end{align*}

(\text{in order of rotation})

(\text{Euler angles})

\text{Euler Rotation Transformations}
associated Euler transformation angles, \( \alpha, \beta, \gamma \), with respect to the axes.

The Euler transformation matrix, expressed in elemental form, for a 3-1-3 Euler rotation is

\[
\begin{bmatrix}
-\cos \beta & \sin \beta & 0 \\
-\sin \beta & -\cos \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

with associated Euler angles \( \alpha, \beta, \gamma \), is

\[
A = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \beta & \sin \beta & 0 \\
-\sin \beta & -\cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\[(\theta) V_e (\phi) V_1 (\lambda) V_z = V_{123}\]

The well-known examples of Tait-Bryan rotation sequences and associated Tait-Bryan angles are:

<table>
<thead>
<tr>
<th>Order</th>
<th>Tait-Bryan Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2-3</td>
<td>(\alpha, \beta, \gamma)</td>
</tr>
<tr>
<td>2-1-3</td>
<td>(\alpha, \beta, \gamma)</td>
</tr>
<tr>
<td>2-3-1</td>
<td>(\alpha, \beta, \gamma)</td>
</tr>
<tr>
<td>3-1-2</td>
<td>(\alpha, \beta, \gamma)</td>
</tr>
<tr>
<td>3-2-1</td>
<td>(\alpha, \beta, \gamma)</td>
</tr>
</tbody>
</table>

(\text{in order of rotation})

Transformation is:

and yaw (\(\psi\)), pitch (\(\theta\)), and lead-lag (\(\phi\)) changes of a helicopter's rotor blade, and 2.) the roll (\(\zeta\)), pitch
\[ T_M = f([\mathbf{A}], [\mathbf{R_S}]) \]

process can be represented by specified rotation sequence \([\mathbf{R_S}]\) and its associated specific rotation angles \([\mathbf{A}])\). In Section 2.1 and its subsections, the transformation matrix \([\mathbf{T_M}]\) was defined using

\[
\begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
321A \equiv A(\theta)A(\phi)A(\alpha) \equiv \left[
\begin{array}{ccc}
\cos(\phi) & \sin(\phi) & 0 \\
-\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{array}
\right]
\]

sequence with associated Tait-Bryan angles \(\alpha, \beta, \gamma\), is as follows:

The Tait-Bryan transformation matrix, expressed in elemental form, for a 1-2-3 Tait-Bryan

\[
321A \equiv A(\phi)A(\theta)A(\alpha) \equiv
\]

The second example, the associated Tait-Bryan transformation is

For the second example, the associated Tait-Bryan transformation is
-21-

\[ \angle \phi \in [0, 360^\circ) \]

\( \phi = 360^\circ - \Phi \)

\( \Phi \in [0, 180^\circ) \)

\( \Phi = \phi \)

where \( \phi \) is specified by Equation (13), since an angle associated with the real \( \lambda = 1 \) eigenvalue of the associated transformation matrix, with the single associated with the real \( \lambda = 1 \) eigenvalue of the associated transformation matrix, 

is fixed axes. This axis of rotation is the eigenvector of the associated transformation matrix which can be characterized uniquely by a single rotation. 

In accordance with Euler's Theorem (see Property 6 in Section 4.1), any rotation 

2.2.1 Single Rotation

\[ \text{where } p \text{ and } q \text{ denote pseudo-inverse} \]

\[ \mathbf{R} = \mathbf{A} \mathbf{B} \]

The problem can be represented by the rotation matrix \( [\mathbf{R}] \) given the rotation sequence \( [\mathbf{R}, \mathbf{S}] \) is addressed in this Section and its subsections.
both the sine and cosine of each of the two angles \( \phi \), \( \psi \) are specific elements of the transformation matrices. The inverse (or transpose), as defined by equations (39) through (44), for these special cases.

The axes of rotation will be coincident with coordinate axes only if the transformation matrix is infinite. The axes of rotation are, however, not necessarily coincident with any of the coordinate axes.

In general, the set of possible two angle rotations which will yield a specific transformation as applicable, and \( \phi = \alpha \), \( \gamma \), or \( \beta \) as appropriate, where \( \sin \phi \) and \( \cos \phi \) are elements from the matrices specified in equations (30), (33), (33), or (36). In this case, the rotation angle is

\[
\phi = \text{Arc}(S, \phi, C, \phi)
\]

Three elemental transformation matrices, or their inverses (or transposes), as defined by equations (39) through (44), will only be coincident with any of the coordinate axes unless they are a transformation matrix of the form of one of the transformation matrices associated with \( \alpha = 1 \). In general, not coincident.
The orthogonality requirement (i.e. Property 2 in Section 2.1) the appropriate diagonal must be identical to zero.

2.2.3 Three Rotations

\[
\begin{align*}
\psi &= \arccos (\sin \phi, \cos \phi) \\
\phi &= \arccos (\sin \psi, \cos \psi)
\end{align*}
\]

where \( \phi \) and \( \psi \) = \( \alpha, \beta, \gamma \) as applicable.

Elements of the transformation matrix. Therefore, accordingly, the two angles \( \phi \) and \( \psi \) can be obtained directly from the appropriate matrix.
The absence of matrix elements whose values are zero and/or one. These conditions are summaries be selected and the corresponding rotation angles determined regardless of the presence and/or sequence with rotations about axes which are coincident with associated co-ordinate system axes of the orthogonality condition. This means that for a specified transformation matrix, a three are, however, no special requirements on the elements of the transformation matrix other than cases, the axes of rotation are not necessarily coincident with any of the co-ordinate axes.

For three angle rotations, there is also an infinity of three angles for in addition to satisfying the orthogonality requirement, the appropriate off-diagonal elements must be zero. For three angle rotations, the transformation matrices defined by Equations (39) through (44), This latter requirement means that only if the transformation matrix is of the form of one of the six elemental rotations axes of rotation can correspond to, and/or define, a specific transformation matrix, and the corresponding two axes of rotation would be coincident with associated co-ordinate system rotation sets which correspond to, and/or define, a specific transformation matrix, and to Subsection 2.1.2, Two Rotations, it was noted that in general there was an infinity of two.
Equation (31). The required angles are:

The Euler rotation angles can easily be determined from the elements of the associated Euler transformation matrix. As an example, consider the 313A Euler transformation matrix defined by:

<table>
<thead>
<tr>
<th>Yes</th>
<th>None</th>
<th>None</th>
<th>Yes</th>
<th>Infinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>None</td>
<td>One</td>
<td>Yes</td>
<td>Infinite</td>
</tr>
<tr>
<td>No</td>
<td>One</td>
<td>Four</td>
<td>Yes</td>
<td>2</td>
</tr>
</tbody>
</table>

Three Angle Rotation

Sequence 1: Orthogonal

Sequence 2: Orthogonal

Sequence 3: Orthogonal

Sequence 4: Orthogonal

Conditions for Axes Coincidence

Ortho-Angle Requirement

Rotation Sets

Principal Angle
\[ A_{31} = 31.3 \text{ A} \]
\[ A_{32} = 31.3 \text{ A} \]
\[ A_{33} = 31.3 \text{ A} \]

where

\[
\begin{bmatrix}
\frac{\sin \phi}{A_{31}}, & \frac{\cos \phi}{A_{31}}
\end{bmatrix}
\]

\[
\rho = \text{Arc} \left( \frac{\sin \phi}{A_{32}} \right)
\]
\[
\alpha = \text{Arc} \left( -\frac{\sin \phi}{A_{33}} \right)
\]

Select one of these values (i.e., either \( \rho \) or \( \alpha \)) and proceed.

\[
\begin{cases}
\rho = 360 - \rho, & \text{if } \rho < 180, 360 \text{, and } \rho_2 \\
\rho = 0, 180 \text{, and } \rho_2
\end{cases}
\]

There are two principal values of \( \rho \), \( \rho_1 \), and \( \rho_2 \).

\[
\rho = \rho_2 - (A.33)
\]
A_{32} = 313A(3,2) = -\cos \alpha \sin \beta \\
A_{33} = 313A(3,3) = \cos \beta \\

Equations (70) and (71) have singularities when 
\[ \beta = m(180^\circ) \text{ for } m \in \mathbb{Z} \text{ integers} \]

2.2.3.2 Tait-Bryan Angles

The Tait-Bryan rotation angles can easily be determined from the elements of the associated Tait-Bryan transformation matrix. As an example, consider the 321 Tait-Bryan transformation matrix defined

\[ \beta = \sin^{-1}(A_{31}) \]

Equation (61). The required angles are:

There are two principal values of \( \beta \), \( \beta_1 \) and \( \beta_2 \).

\[ \beta = \{ \beta_1 \in [270^\circ, 90^\circ], \beta_2 = 180^\circ - \beta_1 \in [90^\circ, 270^\circ] \} \]

Select one of these values (i.e., either \( \beta_1 \) or \( \beta_2 \)) and proceed.
Equations (74) and (75) have singularities when \( \beta = (2m+1)(180) \) for \( m \in \mathbb{Z} \), integers.

\[
A_{31} = 32! A(3,1) = \sin \beta
\]

\[
A_{32} = 32! A(3,2) = -\cos \beta \sin \gamma
\]

\[
A_{21} = 32! A(2,1) = -\cos \beta \sin \gamma
\]

\[
A_{11} = 32! A(1,1) = \cos \beta \sin \gamma
\]

\[
A_{33} = 32! A(3,3) = \cos \alpha \cos \beta
\]

\[
Y = \arctan \left( \frac{A_{21}}{A_{22}}, \frac{A_{11}}{A_{33}} \right)
\]

\[
\alpha = \arccos \left( \frac{-A_{32}}{\cos \beta}, \frac{A_{33}}{\cos \beta} \right)
\]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[E_3 \text{ is the (3x3) identity matrix.}\]

\[
0 = \begin{vmatrix}
d_{13} & d_{23} & (d_{33} - \chi) \\
d_{23} & d_{13} & (d_{22} - \chi) \\
d_{33} & d_{22} & d_{11} - \chi
\end{vmatrix}
\]

Let \((\theta^3 - \chi E_3) = 0\) or equivalently

\[\det(\theta^3 - \chi E_3) = 0\]

The characteristic equation is defined

\[
\begin{pmatrix}
d_{13} & d_{33} & d_{23} \\
d_{23} & d_{13} & d_{33} \\
d_{33} & d_{23} & d_{11}
\end{pmatrix}
= \begin{pmatrix}
d_{33} & d_{23} & d_{13} \\
d_{23} & d_{13} & d_{33} \\
d_{13} & d_{33} & d_{23}
\end{pmatrix}
= \theta^3
\]

If given a real symmetric (3x3) rank two tensor \(G\)

\[\text{Rank Two Tensor}\]

2.3 Basic Properties of the Characteristic Equation of a Real Symmetric (3x3)
\[ I_3, I_2, \text{ and } I_1 \text{ are invariants under admissible transformations of } \mathcal{G}. \]

\[
\begin{align*}
I_3 &= d_{33}^2 + 2d_{13}d_{33} - d_{11}(d_{33})^2 - d_{22}(d_{13})^2 - d_{33}(d_{12})^2 \\
I_2 &= \det(\mathcal{G})^3 \\
I_1 &= d_{22}d_{33} - \det(\mathcal{G})(d_{33}^2 + d_{11}d_{22} - d_{13}^2) \\
\end{align*}
\]

\[
\begin{align*}
I_1 &= \text{Tr}(\mathcal{G})^2 \det(\mathcal{G}) \\
I_2 &= d_{11}d_{22} + d_{33}d_{13} - d_{13}^2 \\
I_3 &= d_{11}d_{22}d_{33} - d_{11}d_{22}d_{33} + 2d_{11}d_{22}d_{33} - d_{13}^2 - d_{22}d_{33}^2 \\
\end{align*}
\]

Characterstic Equation

\[
(\chi - I_3)^2 + I_2\chi - I_3 = 0
\]

Equation (77) can be expressed as
\[ \frac{2 \pi}{1} - I_1^2 - 2 \pi I_1 \] \[ \frac{2 \pi}{1} = 4 \] \[ \frac{5}{1} \] \[ \frac{3}{1} \] \[ \frac{2}{1} \]

Equations (79), (80), and (81), respectively. Then \( A, f, \) and \( \phi \) are defined according to:

First, \( I_1, I_2, \) and \( I_3 \) are evaluated to determine the values of \( \chi_1, \chi_2, \) and \( \chi_3. \)

Since the roots of Equation (78) are known to be real, the following specific cubic equation solution holds:

\[ \chi_3 = \chi_1 ! \chi_2 ! \chi_3 \]
\[ \chi_2 = \chi_1 ! \chi_2 ! \chi_3 \]
\[ \chi_1 = \chi_1 ! \chi_2 ! \chi_3 \]

or equivalently, the invariants \( I_1, I_2, \) and \( I_3 \) of Equation (78) become:

\[ 0 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \]

Since \( D \) is real symmetric, the roots \( x_1, x_2, \) and \( x_3 \) of Equation (78) are real.
And so \( \phi \) and \( \lambda_0 \) expressed by Equations (88) and (89) are real.

The inequality expressed by Equation (90) can only be true if

\[
\begin{align*}
\frac{4}{9} a^2 + 3 \lambda_0^2 &< 0 \\
\frac{4}{27} a^2 + \lambda_0^2 &< 0
\end{align*}
\]

This cubic equation solution process is used when the roots are real or, equivalently, when

\[
\lambda_0 = \frac{1}{3} \left( \frac{8}{27 \sqrt[3]{v}} \right)^{1/2} \left( \frac{4}{27} a^2 + \lambda_0^2 \right)^{1/2}
\]

for \( k = 1, 2, 3 \)

Then

\[
\phi = \cos^{-1} \left( \frac{8 \sin k \lambda_0}{\sqrt{-\frac{27 k^2}{4}}^{1/2}} \right)
\]
such that $D$ is diagonal. Then determine the Tait-Bryan angles corresponding to a specified Tait-Bryan rotation sequence.

where $D = \Theta$

and a transformation matrix $A_1$ from coordinate system 1 to coordinate system 2, so that

$$
\begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{bmatrix} =
\begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{bmatrix} = \Theta
$$

The problem addressed in this section is to determine the Tait-Bryan angles, corresponding to a given rotation sequence.

Rank Two Tensor 2.4 Determination of Tait-Bryan Angles Which Will Diagonalize a Real Symmetric...
\[ -34 \]

\[
\begin{align*}
\mathbf{A}_1 & = \mathbf{A}_3 \mathbf{A}_1^T \\
\mathbf{A}_2 & = \mathbf{A}_3 \mathbf{A}_2^T \\
\mathbf{A}_1 \mathbf{A}_1^T & = \mathbf{D}
\end{align*}
\]

Equation (96) can be written

\[
\begin{bmatrix} \mathbf{A}_N \\ \mathbf{A}_{N-1} \\ \vdots \\ \mathbf{A}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^T \\ \mathbf{A}_2^T \\ \vdots \\ \mathbf{A}_N^T \end{bmatrix}
\]

Nothing that

\[
\begin{align*}
\mathbf{T} & = \mathbf{T}^{-1} = \mathbf{T}^T \mathbf{B} \mathbf{B}^T \\
\mathbf{T} \mathbf{B} \mathbf{B}^T & = \mathbf{B} \mathbf{B}^T \mathbf{T}
\end{align*}
\]

where \( \mathbf{B} \) is the Felt-Bryan transformation matrix. Diagonalize \( \mathbf{B} \) and \( \mathbf{D} \) are the ass

\[
\begin{align*}
\mathbf{A}_1 & = \mathbf{A}_3 \mathbf{A}_1^T \\
\mathbf{A}_2 & = \mathbf{A}_3 \mathbf{A}_2^T \\
\mathbf{A}_1 \mathbf{A}_1^T & = \mathbf{D}
\end{align*}
\]

and

\[
\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3 \mathbf{A}_1^T \\ \mathbf{A}_3 \mathbf{A}_2^T \\ \mathbf{A}_3 \mathbf{A}_1^T \end{bmatrix}
\]

Then

\[
\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3 \mathbf{A}_1^T \\ \mathbf{A}_3 \mathbf{A}_2^T \\ \mathbf{A}_3 \mathbf{A}_1^T \end{bmatrix}
\]

As an example, consider the 1-2-3 Felt-Bryan rotation sequence for this development. The corre

\[
\begin{bmatrix} \mathbf{D} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3 \mathbf{A}_1^T \\ \mathbf{A}_3 \mathbf{A}_2^T \\ \mathbf{A}_3 \mathbf{A}_1^T \end{bmatrix}
\]
\[ \sin \theta = \text{dir CEC} - \text{dir SGN CEC} + \text{dir SGN CEC} + \text{dir SGN CEC} \]  

\[ \cos \theta = \text{dir SGN CEC} - \text{dir SGN CEC} + \text{dir SGN CEC} + \text{dir SGN CEC} \]  

\[ \text{corresponding scalar equation} \]

\[ \frac{\text{equation (100)}}{\text{element of}} \]

where the relevant elements of equation (100) (i.e., the scalar equations) are

1) \[ \mathbf{D}(x) \mathbf{V}_1 \mathbf{D}(y) \mathbf{V}_2 \mathbf{D}(z) \mathbf{V}_3 \mathbf{D}(x) = \mathbf{D}(x) \mathbf{V}_1 \mathbf{D}(y) \mathbf{V}_2 \mathbf{D}(z) \mathbf{V}_3 \mathbf{D}(x) \]

2) \[ \mathbf{D} \mathbf{V}_1 \mathbf{D} = \mathbf{V}_i \mathbf{V}_i \]

Postmultiplication of equation (98) by \( \mathbf{V}_i \mathbf{V}_i \) yields

This equation, the following procedure was selected.

After considerable experimentation in the manipulation of \( d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}, d_{33} \), \( x_1, x_2, x_3 \), and \( \lambda \), \( \mathbf{A} \) can be expressed as an explicit scalar function of simultaneous scalar equations from which \( \mathbf{A} \) and \( \mathbf{R} \) can be expressed as explicit scalar functions of \( \mathbf{D} \).

The solution plan is to manipulate equation (98) and search for elements which yield a tractable solution.
Equation (10a) has two principal solutions, \(\alpha_1\) and \(\alpha_2\), of interest. If \(\alpha_1\) is the solution which

\[
\tan \alpha = \frac{d_{12} - (d_{22} - x_3)(d_{11} - x_3)}{d_{12}d_{13} - d_{13}d_{11}(d_{11} - x_3)}
\]

From which

\[
\frac{d_{12} - (d_{22} - x_3)\sin \alpha - d_{23}\cos \alpha}{(d_{22} - x_3)\sin \alpha - d_{23}\cos \alpha} = \frac{x_3 - d_{12} \sin \alpha}{d_{12} \sin \alpha - d_{13} \cos \alpha}
\]

Dividing Equation (10a) by Equation (10c) yields respectively.

Equation (10c) and (10d) can be rearranged to yield
Finally, an expression for \( \tan \theta \) can be obtained from Equation (101), i.e., \( 3\tan \theta = \tan \phi \) implies \( \phi = (\phi_1 + 180^\circ \text{ or } 180^\circ, 360^\circ) \).

Then

\[
\phi \in (0^\circ, 180^\circ)
\]

Assuming either \( \phi_1 \) or \( \phi_2 \), Equation (110) has two principal solutions, \( \phi_1 \) and \( \phi_2 \), of interest.

Equation (124) can be solved for \( \tan \phi \). Specifically,

\[
\frac{x_3 - x_1}{x_2 - x_1} \tan \phi = \frac{z_1 - z_2}{z_3 - z_2}
\]

Then

\[
\phi = \phi_1 + 180^\circ \text{ or } 180^\circ, 360^\circ
\]
that the tensor characterizing an axial symmetry about the third axis.

the relation angle $\theta$ associated with $x_1$ and $x_2$ is indeterminate or not conceptually significant

reduces to the indeterminate $0.$ Form not resolvable with hospital rule. This is a

If repeated eigenvalues will now be examined. It can be shown, for example, that if $\lambda_1 = \lambda_2 = \lambda_3$, then

Up to this point, it has been assumed that the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ were all distinct.

Then

$$\phi = \theta + 180^\circ \in [180^\circ, 360^\circ)$$

$$\phi \in [0^\circ, 180^\circ)$$

The quadrant is that if $\phi$ principal solutions $\lambda_1$ and $\lambda_2$ of interest. If $\lambda_3$ is the solution which lies in the first or second

assuming specific solution values of $\lambda$ and $\phi$ (i.e., $\lambda_1$, $\lambda_2$ or $\phi$ and $\phi$, or $\phi$ and $\phi$), equation (113) leads

$$\tan \phi = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$$
For convenience, the elements of Eq. (116) (i.e., the elements of $\theta$) are listed below.

\[
\begin{align*}
& d_1 = d_{12} = d_{13} = -A_{21} \alpha_2 \kappa_2 \kappa_1 + A_{12} \kappa_1 \kappa_2 r_2 - A_2 \kappa_2 \kappa_1 \kappa_2 r_2 + A_1 \kappa_1 \kappa_2 \kappa_2 r_2 \\
& d_{21} = d_{22} = d_{23} = -A_{12} \alpha_2 \kappa_2 \kappa_1 + A_{11} \kappa_1 \kappa_2 r_2 - A_1 \kappa_1 \kappa_2 \kappa_2 r_2 + A_2 \kappa_2 \kappa_1 \kappa_2 r_2 \\
& d_{31} = d_{32} = d_{33} = A_{21} \alpha_2 \kappa_2 \kappa_1 + A_{22} \kappa_1 \kappa_2 r_2 - A_2 \kappa_2 \kappa_1 \kappa_2 r_2 + A_1 \kappa_1 \kappa_2 \kappa_2 r_2
\end{align*}
\]

For the effect of repeated eigenvalues, refer to Eq. (98). Which is easily derived from Eq. (116) or (99).
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\Rightarrow A = E
\]

and

3.

For \( x_1 = x_2 = x_3 \)

\[
\begin{bmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\Rightarrow \begin{bmatrix}
d_{x_1} d_{x_2} d_{x_3} \\
d_{x_1} d_{x_2} d_{x_3} \\
d_{x_1} d_{x_2} d_{x_3}
\end{bmatrix}
\] or equivalently

\[
D = \Theta
\]

First, consider the case where all the eigenvalues are equal. Indeed, this case makes sense.

\[
\begin{cases}
\frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_3 \cos \theta = \frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_2 \cos \theta = \frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_1 \cos \theta
\end{cases}
\]

\[
d_{x_3} = \frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_3 \cos \theta = \frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_2 \cos \theta = \frac{d_{x_1} d_{x_2} d_{x_3}}{d_{x_1} d_{x_2} d_{x_3}} + x_1 \cos \theta
\]
\[ d_{33} = \lambda (\omega_{2} \omega_{3} + \omega_{3} \omega_{1}) + \lambda_{3} \omega_{1} \omega_{2} \omega_{3} \]

\[ d_{23} = \mu_{2} = \lambda (\omega_{2} \omega_{3} + \omega_{3} \omega_{1}) + \lambda_{3} \omega_{1} \omega_{2} \omega_{3} \]

\[ d_{13} = \nu_{3} = -\lambda (\omega_{2} \omega_{3} + \omega_{3} \omega_{1}) + \lambda_{3} \omega_{1} \omega_{2} \omega_{3} \]

\[ d_{12} = \omega_{1} = \lambda (\omega_{2} \omega_{3} + \omega_{3} \omega_{1}) + \lambda_{3} \omega_{1} \omega_{2} \omega_{3} \]

Next, consider the case where \( \lambda = \lambda_{1} \neq \lambda_{2} = \lambda_{3} \neq \lambda_{4} \neq \lambda_{5} \neq \lambda_{6} \). It was already noted that in this case, \( \phi \) is the same regardless of the choice of coordinates. The resulting elements of \( \phi \) become indeterminate. This result is arrived at by first substituting 0 into the symmetric about a point, and that their representation in the same region. Under consideration have symmetrical about a point, and that their representation is the same region. This suggests that the tensor changes values, which are indeterminate, are not consequential. This suggests that the tensor changes values, which are indeterminate, are not consequential.
For $x_1 = x_2 \neq x_3$

$$\tan \alpha = -\frac{d_1}{d_2}$$

Equation (126) divided by Equation (127) yields,

Simplified expressions for $a$ and $b$ can be obtained from Equations (125), (126), and (127).

$\alpha \in \left[0^\circ, 180^\circ\right)$

The first or second quadrant, that is if

Equation (131) has the principal solutions, $x_1$ and $x_2$, of interest. If $x_1$ is the solution which

Stated conclusion that $p = 0$, as expressed by Equation (113), is of the $0/0$ indeterminate form.

Substitution of Equations (125), (126), and (127) into Equation (113) leads to the previous

by Equation (116) is not dependent on $p$ in this case.

This means that the transformation does not appear from the transformation when $x_1 = x_2 \neq x_3$. This means that the third rotation angle, $\gamma$, complete

From Equations (125) through (130), it can be seen that the third rotation angle, $\gamma$, complete
With the appropriate selection of Equation (125) or (136), $\theta$ is expressed

$$\sin 2\theta = \frac{(\chi - \chi^3) \cos \alpha}{-2 \delta_{13}}$$

written as

If $\alpha \neq (2\pi + 1)(180^\circ)$, for $m \in \mathbb{Z}$, or equivalently if $\cos \alpha \neq 0$, Equation (127) can

$$\sin 2\theta = \frac{(\chi - \chi^3) \sin \alpha}{2\delta_{12}}$$

written as

If $\alpha \neq n(180^\circ)$, for $m \in \mathbb{Z}$, or equivalently if $\sin \alpha \neq 0$, Equation (126) can

$$\cos 2\theta = \frac{\chi - \chi^3}{2\delta_{11} - \chi - \chi^3}$$

Rearrangement of Equation (125) with appropriate trigonometric substitution yields

$$\alpha_2 = \alpha_1 + 180^\circ \times [180^\circ, 360^\circ]$$

Then
The procedure described by Equations (125) through (135) for cases where \( x_3 \neq x \), or \( x_3 = x \), \( x_1 \neq x \), or \( x_1 = x \), \( x_2 \neq x \), or \( x_2 = x \), where \( x_3 \neq x \), can be determined by deriving from the transformation to the cases \( x_1 \neq x_2 \). Where a rotation angle was considered exhibited by Equations (125) through (135) for \( x_1 \neq x_2 \), where \( x_3 \neq x \), there was a slight simplification to Equations (127) through (133) for \( x_1 \neq x_2 \). Although insignificant simplification of the investigation of the effect of repeated eigenvalues for the cases \( x_2 = x \) and \( x_3 = x \), and \( x_1 = x \), \( x_2 = x \) cases employed t

\[ \beta = \beta_1 + 180^\circ \in [180^\circ, 360^\circ) \]

Then

\[ \beta \in [0^\circ, 180^\circ) \]

in the first or second quadrant, that is if Equation (137) has the principal solutions, \( \beta \) and \( \beta_2 \), of interest. If \( \beta \) is the solution which

\[ \beta = \frac{1}{2} \arccos (\sin 2\beta, \cos 2\beta) \]
For the other Fert-Bryan rotation sequences (i.e., 2-3-1, 3-1-2, 3-2-1, and 2-1-3).

3. If \( x_1 = x_2 = x_3 \), \( a \), \( b \), and \( f \) are all indeterminate.

(131) through (139) and \( f \) is indeterminate.

2. If \( x_1 = x_2 \neq x_3 \), \( a \) and \( f \) can be determined by the procedure described by Eqn (115).

1. If the eigenvalues are all distinct (i.e., \( f_1 \neq f_2 \neq f_3 \)), or if \( f_1 = f_2 \neq f_3 \), or \( f_1 = f_2 = f_3 \), or \( f_1 \neq f_2 \neq f_3 \).

For a 1-2-3 Fert-Bryan rotation sequence such as that defined by Eqn (95).

\[ 3\pi A = 3\pi A(A_1)(A_2)(A_3) \]

3.0 RESULTS
The process to define the tail-Bryan angles required to diagonalise a real symmetric (\(3 \times 3\)) rank-two tensor was identified. This process is applicable to commonly encountered tail-Bryan angles from a transformation matrix.

Tail-Bryan angles from a transformation matrix.

Tail-Bryan angles, Transformation matrix:

\[ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \]

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11. Supplementary Notes

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14. Abstract (Maximum 200 words)

The process to define the tail-Bryan angles required to diagonalise a real symmetric (\(3 \times 3\)) rank-two tensor was identified. This process is applicable to commonly encountered tail-Bryan angles from a transformation matrix.

Appendix A

Real Symmetric (\(3 \times 3\)) Rank Two Tensor

Tail-Bryan Angles Required For Diagonalization Of A