NASA Technical Memorandum 110389

Tait-Bryan Angles Required for Diagonalisation of a Real Symmetric (3 × 3) Rank Two Tensor

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April 1996

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ABSTRACT

Tait-Bryan Angles
Required for Diagonalisation of a
Real Symmetric (3 x 3) Rank Two Tensor

The process to define the Tait-Bryan angles required to diagonalise a real symmetric (3 x 3) rank two tensor was identified. This process is applicable to commonly encountered tensors such as inertia tensors, rank two (3 x 3) covariance tensors, and quadratic coefficient tensors. Tait-Bryan angles, which are any permutation of 1-2-3 rotation angles, are used to define the pitch, flap, and lead-lag angles in helicopter rotor systems. The Tait-Bryan angles differ from the more familiar Euler angles which are any permutation of 3-1-3 rotation angles.

CONTENTS

NOMENCLATURE

A — linear orthonormal transformation (matrix) from one co-ordinate system to an

Pre-superscripts

$\cdots kji$ — elemental axes of rotation in sequence order from right to left for rotation sequence $ijk\cdots$ (e.g., the transformation matrix for an rotation would be written as ${}^{kji}A$

Post-superscripts

k — denotes an exponent (the corresponding transformation matrix is writte A^k) with the following exceptions.

ji — a pair of integers which denote that the transformation is from co-o system i to co-ordinate system j (the corresponding transformatio is writen as A^{ji})

-1 — denotes the inverse (the corresponding transformation matrix is written

T — denotes the transpose (the corresponding transformation matrix is writt

(k) — denotes the k-th column partition of A (the corresponding column v is written as $A^{(k)}$)

Post-subscripts

ij — a pair of integers which denote the (i,j)th element of A (the element is w
as A_{ij}) Note, $A_{ij} \equiv A(i,j)$

Arguments

Φ — a single angular argument, used only for single elemental rotations, deno
rotation angle.

i,j — a pair of integers which denote the (i,j)th element of A (the element is
as $A(i,j)$) Note, $A(i,j) \equiv A_{ij}$

Example $\quad {}^{2321}A^T(2,1) \equiv {}^{2321}A^T_{21} \equiv \left[{}^{2}A\,{}^{3}A\,{}^{2}A\,{}^{1}A\right]^T(2,1) \equiv \left[{}^{2}A\,{}^{3}A\,{}^{2}A\,{}^{1}A\right]^T_{21} \equiv$ (2,1)st element of
transpose of a 1-2-3-2 rotation sequence transformation matri

a — an intermediate value comprised of combinations of I_k in the solution of the cubic chara
equation of a real symmetric (3X3) rank two tensor.

b — an intermediate value comprised of combinations of I_k in the solution of the cubic chara
equation of a real symmetric (3x3) rank two tensor.

$\mathcal{D}$ — a real symmetric (3x3) rank two tensor

D — diagonal form of $\mathcal{D}$

d_{ij}	(i,j)th element of $\mathcal{D}$
E_3	(3×3) identity matrix
F	general rank two tensor
F^k	general rank two tensor expressed in co-ordinate system k
F_{ij}	(i,j)th element of F
f	general function
I_k	kth invariant of the characteristic equation of a real symmetric (3×3) rank two
i	orbital inclination
i	element position index
i	rotation index
j	imaginary number base $= \sqrt{-1}$
j	element position index
j	rotation index
k	co-ordinate system number denoting co-ordinate system k
k	eigenvalue solution number
k	rotation index

n	an integer
PI	post superscript of f denoting "Pseudo-Inverse"
[RA]	Rotation Angles
[RS]	Rotation Sequence
[TM]	Transformation Matrix
V	general 3-Vector
V^k	general 3-Vector expressed in co-ordinate system k
α	first rotation angle
α	Kronecker δ Symbol index
α	Column vector partition index
α	vehicle pitch angle
β	second rotation angle
β	Kronecker's Symbol index
β	blade flap angle
β	vehicle yaw angle
γ	third rotation angle

$\delta_{\alpha\beta}$	kronecker's Symbol, $\delta_{\alpha\beta} = \begin{cases} 0 & \text{for } \alpha \neq \beta \\ 1 & \text{for } \alpha = \beta \end{cases}$
ζ	blade lead-lag angle
Θ	general rotation angle
θ	blade pitch angle
Λ	vector of eigenvalues (note, <u>not</u> the same as an eigenvector)
λ	general eigenvalue
λ	value of repeated eigenvalues
λ_j	j th eigenvalue
ν_0	periapsis argument
σ	vehicle roll or bank angle
Φ	general rotation angle
Φ	equivalent single rotation angle as specified by Euler's Theorem
ϕ	intermediate angular value comprised of combinations of I_k in the solution of characteristic equation of a real symmetric (3x3) rank two tensor.
Ψ	general rotation angle
Ω	longitude of the ascending node

1.0 INTRODUCTION

Tait-Bryan angles, Euler angles, and other rotation sequences as used herein are speci[illegible] sequences of rotations of a Euclidean Three-Space Co-ordinate System from an initial orientatio[illegible] a new orientation. Their associated transformation matrices provide a simple means to express th[illegible] elements of an invariant characteristic of an entity in a new co-ordinate system in terms of [illegible] elements expressed in the initial original co-ordinate system. Here it is assumed that the invariance [illegible] the characteristic is with respect to the rotations of the co-ordinates under consideration and no[illegible] for example, with respect to time. The subject problem of this, which is a pseudo-inverse of [illegible] above mentioned procedure, is to determine the Tait-Bryan Angles required to diagonalise [illegible] matrix representing the invariant characteristic in a particular co-ordinate system.

2.0 TECHNICAL

Basic co-ordinate system rotations and the definition of their associated transformation matrices are described in Section 2.1 and its subsections. The pseudo-inverse problem of determining rotation angles from specified transformation matrices and rotation sequences is discussed in Section 2.2 and its subsections. Basic properties of a real symmetric (3x3) rank tensor which are relevant to the topics discussed herein are defined in Section 2.3. Finally, metho to determine the Tait-Bryan Angles which will diagonalise a real symmetric (3x3) rank two tensor presented in Section 2.4.

2.1 BASIC CONCEPTS FOR CO-ORDINATE ROTATIONS AND TRANSFORMATIONS

The co-ordinate system transformations presented herein are linear orthonormal transformations the invariant characteristic expressed in one Euclidean three-space co-ordinate system to that exp in another Euclidean three-space co-ordinate system. These transformations are commonly referred to Cosine Matrices. A summary of important properties of these transformations follows.

Let A denote the transformation matrix. Then

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1}$$

PROPERTY 1 - BASIC VECTOR TRANSFORMATION

If V^k is a vector expressed in co-ordinate system k, $k=1,2$

and A^{21} is the transformation matrix from co-ordinate system 1 to co-ordinate system 2

then $V^2 = A^{21} V^1$ (

PROPERTY 2 - ORTHONORMALITY

Column Orthonormality: $A_{\lambda\alpha} A_{\lambda\beta} = \delta_{\alpha\beta}$ (3

Row Orthonormality: $A_{\alpha\lambda} A_{\beta\lambda} = \delta_{\alpha\beta}$ (4

where $\delta_{\alpha\beta} = \begin{Bmatrix} 0 & \text{for } \alpha \neq \beta \\ 1 & \text{for } \alpha = \beta \end{Bmatrix}$ (KRONECKER'S SYMBOL) (5

and the summation convention is assumed.

Equations (3) and (4) expressed in matrix form are

$$A^T A = A A^T = E_3 \tag{6}$$

where E_3 is the (3×3) identity matrix

Equation (6) implies the following very important property.

<u>PROPERTY 3 - TRANSPOSE EQUALS INVERSE</u>

$$\left.\begin{aligned} A^{-1} &= A^T \\ \text{and} \quad A^T &= A^{-1} \end{aligned}\right\} \tag{7}$$

<u>PROPERTY 4 - DETERMINANT of A</u>

$$\text{Det}(A) \equiv 1 \tag{8}$$

PROPERTY 5 - EIGENVALUES of A

Solution of

$$Det(A - \lambda E_3) = 0 \quad \text{(the Eigenvalue Problem)} \qquad (9$$

for λ yields

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \cos\Phi + j \sin\Phi \\ \cos\Phi - j \sin\Phi \end{bmatrix} \qquad (10)$$

where the Characteristic Equation is

$$\lambda^3 - [Tr(A)]\,\lambda^2 + [Tr(A)]\,\lambda - 1 = 0 \quad \text{(Characteristic Equation)} \qquad (11$$

or equivalently

$$\{\lambda - 1\}\{\lambda^2 + [1 - Tr(A)]\,\lambda - 1 = 0 \quad \text{(Characteristic Equation)} \qquad (12$$

and the angle Φ is the equivalent single rotation angle about the eigenvalue associated with $\lambda_1 = 1$ as specified by Euler's Theorem (presented under Property 6)

It should be noted that Λ is not, in general, an eigenvector; it is a vector of eigenva
(scalar) solutions to Equation (9)

PROPERTY 6 – EULER'S THEOREM and the EQUIVALENT SINGLE ROTATION ANGLE

EULER'S THEOREM Any sequence of co-ordinate transformations can be characterised uniqu
by a unique eigenvector U and an equivalent single rotation angle Φ about it, accordi

$$\cos \Phi = \tfrac{1}{2}\left[\mathrm{Tr}(A) - 1\right] \qquad (13)$$

where the eigenvector U is the eigenvector associated with $\lambda_1 = 1$.

PROPERTY 7 – SUCCESSIVE ROTATIONS

If V^k is a vector expressed in co-ordinate system k, $k = 1, 2, \ldots, N$
and $A^{(k+1)k}$ is the transformation matrix from co-ordinate system k to co-ordinate system

then $$V^N = A^{N(N-1)} A^{(N-1)(N-2)} \cdots A^{32} A^{21} V^1 \qquad (14)$$

and the single equivalent transformation matrix is

$$A^{N1} = A^{N(N-1)} A^{(N-1)(N-2)} \cdots A^{32} A^{21} \tag{15}$$

which means that Equation (14) can be written as

$$V^{N} = A^{N1} V^{1} \tag{16}$$

PROPERTY 8 - TRANSFORMATION OF RANK TWO TENSORS

The inverse transformation of the transformation defined by Equation (2) is

$$V^{1} = \left[A^{21}\right]^{-1} V^{2} \tag{17}$$

Noting the property defined by Equation (7), Equation (17) can be expressed as

$$V^{1} = \left[A^{21}\right]^{T} V^{2} \tag{18}$$

If F^{k} is a rank two tensor expressed in Co-ordinate system k, $k = 1, 2$, then the fundamental tensor invariance identity requires that

$$[V^2]^T[F^2][V^2] = [V^1]^T[F^1][V^1] \quad \left\{ \begin{array}{l} \text{Fundamental Tensor} \\ \text{Invariance Identity} \end{array} \right. \tag{19}$$

Noting Equation (18), Equation (19) can be written as

$$[V^2]^T[F^2][V^2] = [(A^{21})^T V^2]^T[F^1][(A^{21})^T V^2] = [V^2]^T[A^{21}][F^1][A^{21}][V^2] \tag{20}$$

From which

$$[F^2] = [A^{21}][F^1][A^{21}]^T \quad \text{Rank Two Tensor Transformation Law} \tag{21}$$

PROPERTY 9 – RANK TWO TENSOR DIAGONALISATION PROBLEM

The rank two tensor diagonalisation problem requires the determination of the transformation matrix or corresponding rotation angles for an assumed rotation sequence, such that F^2 in Equation (21) i diagonal. Premultiplication of Equation (21) by $[A^{21}]^T$ and rearranging yields

$$[F^1][A^{21}]^T - [A^{21}]^T[F^2] = 0 \tag{22}$$

The first term in Equation (22) can be writen in terms of its elements and then the $[A^{21}]^T$ ma
partitioned into column vectors according to:

$$[F^1][A^{21}]^T = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \left[\begin{array}{c|c|c} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{array}\right] = [F^1]\left[\begin{array}{c|c|c} A^{(1)} & A^{(2)} & A^{(3)} \end{array}\right] \qquad (2$$

where $A^{(\alpha)}$, $\alpha = 1, 2, 3$, are column vector partitions of $[A^{21}]^T$ such that

$$A^{(\alpha)} = \begin{bmatrix} A_{\alpha 1} \\ A_{\alpha 2} \\ A_{\alpha 3} \end{bmatrix}, \quad \alpha = 1, 2, 3 \qquad (2$$

The second term of Equation (22) can be rearranged according to:

$$[A^{21}]^T[F^2] = \left[\begin{array}{c|c|c} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{array}\right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \left[\begin{array}{c|c|c} \lambda_1 A^{(1)} & \lambda_2 A^{(2)} & \lambda_3 A^{(3)} \end{array}\right] \qquad (25$$

Equation (22) can now be expressed as

$$[F'][A^{(1)} \vdots A^{(2)} \vdots A^{(3)}] - [\lambda_1 A^{(1)} \vdots \lambda_2 A^{(2)} \vdots \lambda_3 A^{(3)}] = 0 \qquad (2$$

which is equivalent to three equations of the form

$$[F'] A^{(\alpha)} - \lambda_\alpha A^{(\alpha)} = 0 \quad \text{for } \alpha = 1, 2, 3 \quad ; \alpha \text{ NOT SUMMED} \qquad (2$$

Equation (27) can now be written in the standard eigenvalue problem form

$$(F' - \lambda_\alpha E_3) A^{(\alpha)} = 0 \quad \text{for } \alpha = 1, 2, 3 \quad ; \alpha \text{ NOT SUMMED} \qquad (2$$

The eigenvalues λ of F' are determined by solving the characteristic equation of F' (See Section 2.3)

$$\mathrm{Det}(F' - \lambda E_3) = 0 \quad \text{Characteristic Equation of } F' \qquad (2$$

where α has been dropped since the form of Equation (28) with respect to λ_α is independe of the specific value of α.

An important implication of Equations (28) and (29) is that the diagonalisation of a rank tw tensor is an eigenvalue problem with the diagonal elements of the diagonalised form being the eigenvalues of F (note, the superscript can be dropped because the result is independent of the co-ordinate system F is expressed in), and that this result is not at all dependent on the assumed rotation sequence used to define the diagonalising transformation matrix.

The remainder of this section is divided into two subsections. The definition of the elemen transformation matrices is presented in Subsection 2.1.1, and the definition of the transformatio matrix from a specified rotation sequence and corresponding rotation angles is discussed i Subsection 2.1.2.

2.1.1. ELEMENTAL TRANSFORMATION MATRICES

Assuming a Euclidean three-space co-ordinate system whose axes are numbered 1, 2, an an elemental transformation is a linear orthonormal transformation of co-ordinates from that for the initial co-ordinate system to that specified for a co-ordinate system defined by rotating original co-ordinate system about one of its axes. The right-handed convention is assumed material presented herein. A vector V expressed in the reoriented co-ordinate system, V^2, is related to that expressed in the initial co-ordinate system, V^1, according to Equation (2). A ran two tensor F expressed in the reoriented co-ordinate system, F^2, is related to that expressed in th initial co-ordinate system, F^1, according to Equation (21). For these elemental cases whose rot sequence is a single rotation about one of the co-ordinate axes, the transformation matrices becom

For a rotation α about the first axes, Axes 1

$$A^{21} \equiv {}^1A(\alpha) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \quad (30)$$

then

$$V^2 = {}^1A(\alpha)\, V^1 \quad (31)$$

and

$$F^2 = {}^1A(\alpha)\, F^1\, {}^1A^T(\alpha) \quad (32)$$

For a rotation β about the second axes, Axes 2

$$A^{21} \equiv {}^2A(\beta) \equiv \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \quad (33)$$

then

$$V^2 = {}^2A(\beta)\, V^1 \qquad ($$

and

$$F^2 = {}^2A(\beta)\, F^1\, {}^2A^T(\beta) \qquad ($$

For a rotation γ about the third axes, Axes 3

$$A^{21} \equiv {}^3A(\gamma) \equiv \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad ($$

then

$$V^2 = {}^3A(\gamma)\, V^1 \qquad (3$$

and

$$F^2 = {}^3A(\gamma)\, F^1\, {}^3A^T(\gamma) \qquad (3$$

where

A^{21} is the transformation matrix from the initial co-ordinate system to that obtained by the Element rotation.

${}^kA(\theta)$ indicates that the transformation is an elemental transformation obtained by rotating the initial co-ordinate system through the rotation angle θ about the kth axis.

2.1.2 DETERMINATION OF THE TRANSFORMATION MATRIX CORRESPONDING TO A SPECIFIED ROTATION SEQUENCE AND ASSOCIATED ROTATION ANGLES

Physical characteristics of a system, such as gimbals, hinges, the intersection of orbital and equatorial planes, et cetera, are natural axes for rotating co-ordinate systems and are freque used as such. The pitch, flap, and lead-lag axes of a helicopter's articulated rotor hub is an exam of particular interest. The Successive Rotation Property (Property 7 in Section 2.1) indicates that t total transformation can be comprised of as many different transformations as desired. The resul total transformation matrix, which is used to define the co-ordinates of a vector or rank two te in one co-ordinate system given its co-ordinates in an initial co-ordinate system, is defined accordi Equation (15).

The six possible combinations of two different elemental transformations are presented below

$$
{}^{21}A \equiv {}^{2}A(\beta)\,{}^{1}A(\alpha) \equiv \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix} \quad (
$$

$$ {}^{32}A \equiv {}^{3}A(\gamma)\,{}^{2}A(\beta) \equiv \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\gamma & \sin\gamma & -\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \cos\gamma & \sin\beta\sin\gamma \\ \sin\beta & 0 & \cos\beta \end{bmatrix} $$

$$ {}^{13}A \equiv {}^{1}A(\alpha)\,{}^{3}A(\gamma) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma\cos\alpha & \cos\gamma\cos\alpha & \sin\alpha \\ \sin\gamma\sin\alpha & -\cos\gamma\sin\alpha & \cos\alpha \end{bmatrix} $$

$$ {}^{12}A \equiv {}^{1}A(\alpha)\,{}^{2}A(\beta) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ \sin\beta\sin\alpha & \cos\alpha & \cos\beta\sin\alpha \\ \sin\beta\cos\alpha & -\sin\alpha & \cos\beta\cos\alpha \end{bmatrix} $$

$$ {}^{23}A \equiv {}^{2}A(\beta)\,{}^{3}A(\gamma) \equiv \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\beta & \sin\gamma\cos\beta & -\sin\beta \\ -\sin\gamma & \cos\gamma & 0 \\ \cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{bmatrix} \quad ($$

$$^{31}A \equiv {}^{3}A(\gamma)\,{}^{1}A(\alpha) \equiv \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\gamma & \cos\alpha\sin\gamma & \sin\alpha\sin\gamma \\ -\sin\gamma & \cos\alpha\cos\gamma & \sin\alpha\cos\gamma \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \quad (\text{}$$

where

^{jk}A denotes $^{j}A(\cdot)\,^{k}A(\cdot)$ for $j, k = 1, 2, 3$

Note that, in general, $^{kj}A \neq {}^{jk}A$ and $^{kj}A \neq {}^{jk}A^{T}$ (or $^{jk}A^{-1}$) for $j, k = 1, 2, 3$ but $j \neq k$. Note also that each ^{jk}A has a unique position (j, k) whose element is identically zero. This latte[r] property is useful when solving the pseudo-inverse problem of determining the rotation angles give[n] the transformation matrix and the rotation sequence.

2.1.2.1 Euler Angles

Euler rotation sequences are comprised of three elemental rotations where the first and las[t] rotations are about the same numbered axis and the second rotation is about one of the other tw[o] axes. Euler angles are the rotation angles associated with an Euler rotation sequence. There are six

Euler rotation sequences with associated Euler transformations and Euler angles. These are

Euler Rotation Sequence (in order of rotation)	Euler Transformation	Euler Angles (in order of rotation)
1 - 2 - 1	${}^{121}A \equiv {}^{1}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)$	α, β, γ
1 - 3 - 1	${}^{131}A \equiv {}^{1}A(\gamma)\,{}^{3}A(\beta)\,{}^{1}A(\alpha)$	α, β, γ
2 - 1 - 2	${}^{212}A \equiv {}^{2}A(\gamma)\,{}^{1}A(\beta)\,{}^{2}A(\alpha)$	α, β, γ
2 - 3 - 2	${}^{232}A \equiv {}^{2}A(\gamma)\,{}^{3}A(\beta)\,{}^{2}A(\alpha)$	α, β, γ
3 - 1 - 3	${}^{313}A \equiv {}^{3}A(\gamma)\,{}^{1}A(\gamma)\,{}^{3}A(\alpha)$	α, β, γ
3 - 2 - 3	${}^{323}A \equiv {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{3}A(\alpha)$	α, β, γ

A well known example of an Euler rotation sequence and associated Euler angles is the ang element subset of the classical orbital elements set. In this case, the angular elements are: 1) longitude of the ascending node Ω, 2) the orbital inclination i, and 3) the periapsis argument and the associated Euler transformation is:

$$ {}^{313}A \equiv {}^{3}A(\nu_0)\,{}^{1}A(i)\,{}^{3}A(\Omega) $$

The Euler transformation matrix, expressed in elemental form, for a 3-1-3 Euler rotation s with associated Euler angles α, β, γ, is

$$ {}^{313}A \equiv {}^{3}A(\gamma)\,{}^{1}A(\beta)\,{}^{3}A(\alpha) = \begin{bmatrix} (\cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma) & (\sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma) & (\sin\beta\sin\gamma) \\ (-\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma) & (-\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma) & (\sin\beta\cos\gamma) \\ (\sin\alpha\sin\beta) & (-\cos\alpha\sin\beta) & (\cos\beta) \end{bmatrix} $$

2.1.2.2. TAIT-BRYAN ANGLES

Tait-Bryan rotation sequences are comprised of three elemental rotations such tha of the rotations are about the same numbered axis. Tait-Bryan angles are the rotation angl associated with a Tait-Bryan rotation sequence. There are six Tait-Bryan rotation sequence associated Tait-Bryan transformations and Tait-Bryan angles. These are

Tait-Bryan Rotation Sequence (in order of rotation)	Tait-Bryan Transformation	Tait-Bryan Angles (in order of rotation)
1-2-3	${}^{321}A \equiv {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)$	α, β, γ
2-3-1	${}^{132}A \equiv {}^{1}A(\gamma)\,{}^{3}A(\beta)\,{}^{2}A(\alpha)$	α, β, γ
3-1-2	${}^{213}A \equiv {}^{2}A(\gamma)\,{}^{1}A(\beta)\,{}^{3}A(\alpha)$	α, β, γ
3-2-1	${}^{123}A \equiv {}^{1}A(\gamma)\,{}^{2}A(\beta)\,{}^{3}A(\alpha)$	α, β, γ
1-3-2	${}^{231}A \equiv {}^{2}A(\gamma)\,{}^{3}A(\beta)\,{}^{1}A(\alpha)$	α, β, γ
2-1-3	${}^{312}A \equiv {}^{3}A(\gamma)\,{}^{1}A(\beta)\,{}^{2}A(\alpha)$	α, β, γ

Well known examples of Tait-Bryan rotation sequences and associated Tait-Bryan angles are pitch (θ), flap (β), and lead-lag (ζ) angles of a helicopter's rotor blade, and 2) the roll (σ), pitc and yaw (β) of an aerospace vehicle in flight. For the first example, the associated Tait-Bryan transformation is

$$ {}^{321}A \equiv {}^{3}A(\zeta)\,{}^{2}A(\beta)\,{}^{1}A(\theta) $$

For the second example, the associated Tait-Bryan transformation is

$$ {}^{321}A \equiv {}^{3}A(\beta)\,{}^{2}A(\alpha)\,{}^{1}A(\sigma) $$

The Tait-Bryan transformation matrix, expressed in elemental form, for a 1-2-3 Tait-Bryan sequence with associated Tait-Bryan angles α, β, γ, is

$$ {}^{321}A \equiv {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha) \equiv \begin{bmatrix} (\cos\beta\cos\gamma) & (\sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma) & (-\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma) \\ (-\cos\beta\sin\gamma) & (-\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma) & (\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma) \\ (\sin\beta) & (-\sin\alpha\cos\beta) & (\cos\alpha\cos\beta) \end{bmatrix} $$

2.2 DETERMINATION OF ROTATION ANGLES FROM A SPECIFIED TRANSFORMATION MATRIX AND ROTATION SEC

In Section 2.1 and its subsections, the transformation matrix ([TM]) was defined usin specified rotation sequence ([RS]) and its associated specified rotation angles ([RA]). Th process can be represented by

$$ [TM] = f([RA], [RS]) $$

The pseudo-inverse problem of backing out the rotation angles ([RA]) from a specified transf
matrix ([TM]) given the rotation sequence ([RS]) is addressed in this Section and its Subsections
problem can be represented by

$$[RA] = f^{PI}([TM],[RS])$$

where

PI denotes pseudo-inverse

2.2.1 Single Rotation

In accordance with Euler's Theorem (see Property 6 in Section 2.1), any reorientatio
Euclidean three-space co-ordinate system can be characterised uniquely by a single rotation a
fixed axes. This axis of rotation is the eigenvector of the associated transformation matrix which
associated with the real $\lambda_1 = 1$ eigenvalue of this associated transformation matrix. The single r
angle Φ is specified by Equation (13) where

$$\Phi = \begin{cases} \Phi_1 & \in [0^\circ, 180^\circ) \\ \Phi_2 = 360^\circ - \Phi_1 & \in [180^\circ, 360^\circ) \end{cases}$$

This axis of rotation (i.e., the eigenvector associated with $\lambda_1 = 1$) is, in general, not coincide with any of the co-ordinate system axes unless the transformation matrix is of the form of one o three elemental transformation matrices, or their inverses (or transposes), as defined by Equations (3 (33), or (36). In this case, the rotation angle is

$$\Phi = \mathrm{Arc}\,(\mathrm{Sin}\,\Phi,\ \mathrm{Cos}\,\Phi)$$

where $\mathrm{Sin}\,\Phi$ and $\mathrm{Cos}\,\Phi$ are elements from the matrices specified in Equations (30), (33), or (as appropriate, and $\Phi = \alpha, \beta,$ or γ as appropriate.

2.2.2 Two Rotations

In general, the set of possible two angle rotations which will yield a specific transformatio matrix is infinite. The axes of rotation are, however, not necessarily coincidental with any of t co-ordinate axes. The axes of rotation will be coincidental with co-ordinate axes only if the transfor matrix is of the form of one of the six elemental two rotation angle transformation matrices, their inverses (or transposes), as defined by Equations (39) through (44). For these special cases both the sine and cosine of each of the two angles (Φ, ψ) are specific elements of the transform matrix and, accordingly, the two angles Φ and Ψ can be obtained directly from the appro elements of the transformation matrix.

$$\Phi = \mathrm{Arc}(\mathrm{Sin}\,\Phi, \mathrm{Cos}\,\Phi) \quad (6$$

$$\Psi = \mathrm{Arc}(\mathrm{Sin}\,\Psi, \mathrm{Cos}\,\Psi) \quad (6$$

where Φ and Ψ = α, β, γ as appropriate.

2.2.3 THREE ROTATIONS

In Subsection 2.2.1, Single Rotation, it was noted that in general there were two princip rotations about a fixed axis which correspond to, and/or define a specific transformation m and that this fixed axis of rotation would be coincidental with one of the co-ordinate system only if the transformation matrix is of the form of one of the three elemental transformation m as defined by Equation (30), (33), or (36). This latter requirement means that in addition to satis the orthonormality requirement (i.e., Property 2 in Section 2.1) the appropriate diagonal must be id equal to one and the four appropriate off-diagonal elements must be identically equal to zero

Subsection 2.2.2, Two Rotations, it was noted that in general there was an infinity of two rotation sets which correspond to, and/or define, a specific transformation matrix, and th the corresponding two axes of rotation would be coincidental with associated co-ordinate syste axes only if the transformation matrix is of the form of one of the six elemental two rotation a transformation matrices defined by Equations (39) through (44). This latter requirement means in addition to satisfying the orthonormality requirement, the appropriate off-diagonal element mu identically equal to zero. For three angle rotations, there is also an infinity of three angle rot sets which correspond to, and/or define, a specific transformation matrix and, as in the prev cases, the axes of rotation are not necessarily coincidental with any of the co-ordinate axes. are, however, no special requirements on the elements of the transformation matrix other than sa of the orthonormality condition. This means that for a specified transformation matrix, a three sequence with rotations about axes which are coincidental with associated co-ordinate system axe be selected and the corresponding rotation angles determined regardless of the presence and/or absence of matrix elements whose values are zero and/or one. These conditions are summaris below.

	Number of Possible Principal Angle Rotation Sets	Orthonormality Requirement	Conditions for Axes Coincidence		Freedom to S… Rotation Seq… and axes coin…
			Off-diagonal elements	Diagonal elements	
Single Angle Rotation Sequence	2	Yes	Four Zeros	One One	No
Two Angle Rotation Sequences	Infinite	Yes	One Zero	None	No
Three Angle Rotation Sequences	Infinite	Yes	None	None	Yes

2.2.3.1 EULER ANGLES

The Euler rotation angles can easily be determined from the elements of the associated Euler transformation matrix. As an example, consider the ${}^{313}A$ Euler transformation matrix defined by Equation (52). The required angles are:

$$\beta = \cos^{-1}(A_{33})$$

There are two principal values of β, β_1 and β_2.

$$\beta = \begin{cases} \beta_1 \in [0^\circ, 180^\circ) \\ \beta_2 = 360^\circ - \beta_1 \in [180^\circ, 360^\circ) \end{cases}$$

Select one of these values (i.e., either β_1 or β_2) and proceed.

$$\alpha = \mathrm{Arc}\left[\frac{A_{31}}{\sin\beta}, \frac{-A_{32}}{\sin\beta}\right]$$

and

$$\gamma = \mathrm{Arc}\left[\frac{A_{13}}{\sin\beta}, \frac{A_{23}}{\sin\beta}\right]$$

where

$$A_{13} = {}^{313}A(1,3) = \sin\beta \sin\gamma$$
$$A_{23} = {}^{313}A(2,3) = \sin\beta \cos\gamma$$
$$A_{31} = {}^{313}A(3,1) = \sin\alpha \sin\beta$$

$$A_{32} = {}^{313}A(3,2) = -\cos\alpha \sin\beta$$

$$A_{33} = {}^{313}A(3,3) = \cos\beta$$

Equations (70) and (71) have singularities when $\beta = n(180°)$ for $n \in \{\text{integers}\}$.

2.2.3.2 TAIT-BRYAN ANGLES

The Tait-Bryan rotation angles can easily be determined from the elements of the associated T
transformation matrix. As an example, consider the ${}^{321}A$ Tait-Bryan transformation matrix de
Equation (61). The required angles are:

$$\beta = \sin^{-1}(A_{31})$$

There are two principal values of β, β_1 and β_2.

$$\beta = \begin{cases} \beta_1 \in [270°, 90°) \\ \beta_2 = 180° - \beta_1 \in [90°, 270°) \end{cases}$$

Select one of these values (i.e., either β_1 or β_2) and proceed.

$$\alpha = \mathrm{Arc}\left[\frac{-A_{32}}{\cos\beta}, \frac{A_{33}}{\cos\beta}\right]$$

and

$$\gamma = \mathrm{Arc}\left[\frac{-A_{21}}{\cos\beta}, \frac{A_{11}}{\cos\beta}\right]$$

where

$$A_{11} = {}^{321'}A(1,1) = \cos\beta \cos\gamma$$

$$A_{21} = {}^{321'}A(2,1) = -\cos\beta \sin\gamma$$

$$A_{31} = {}^{321'}A(3,1) = \sin\beta$$

$$A_{32} = {}^{321'}A(3,2) = -\sin\alpha \cos\beta$$

$$A_{33} = {}^{321'}A(3,3) = \cos\alpha \cos\beta$$

Equations (74) and (75) have singularities when $\beta = (2n + 1)(180^\circ)$ for $n \in \{\text{integers}\}$.

2.3 BASIC PROPERTIES OF THE CHARACTERISTIC EQUATION OF A REAL SYMMETRIC (3×3) RANK TWO TENSOR

If given a real symmetric (3×3) rank two tensor $\mathcal{D}$

$$\mathcal{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}$$

The characteristic equation is defined

$$\operatorname{Det}\left(\mathcal{D} - \lambda E_3\right) = 0 \quad \text{or equivalently} \quad \begin{vmatrix} (d_{11} - \lambda) & d_{12} & d_{13} \\ d_{12} & (d_{22} - \lambda) & d_{23} \\ d_{13} & d_{23} & (d_{33} - \lambda) \end{vmatrix} = 0$$

where

E_3 is the (3×3) identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Equation (77) can be expressed as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \qquad \text{Characteristic Equation}$$

where

$$I_1 = Tr(\mathcal{D}) = d_{11} + d_{22} + d_{33}$$

$$\left.\begin{aligned} I_2 &= \begin{vmatrix} d_{22} & d_{23} \\ d_{23} & d_{33} \end{vmatrix} + \begin{vmatrix} d_{33} & d_{13} \\ d_{13} & d_{11} \end{vmatrix} + \begin{vmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{vmatrix} \\ I_2 &= d_{22}d_{33} - (d_{23})^2 + d_{11}d_{33} - (d_{13})^2 + d_{11}d_{22} - (d_{12})^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} I_3 &= Det(\mathcal{D}) = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{vmatrix} \\ I_3 &= d_{11}d_{22}d_{33} + 2d_{12}d_{13}d_{23} - d_{11}(d_{23})^2 - d_{22}(d_{13})^2 - d_{33}(d_{12})^2 \end{aligned}\right\}$$

I_1, I_2, and I_3 are invariants under admissible transformations of $\mathcal{D}$.

Since $\mathcal{D}$ is real symmetric, the roots λ_1, λ_2, and λ_3 of Equation (78) are real. Equation (78 can be expressed in a form in which the constants are defined by the roots; specifically:

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

or equivalently, the invariants I_1, I_2, and I_3 of Equation (78) become

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2$$

$$I_3 = \lambda_1\lambda_2\lambda_3$$

Since the roots of Equation (78) are known to be real, the following specific cubic equation solution p is employed to determine the values of λ_1, λ_2, and λ_3. First I_1, I_2, and I_3 are evaluated Equations (79), (80), and (81), respectively. Then a, b, and ϕ are defined according to:

$$a = \frac{1}{3}\left[3I_2 - I_1^2\right]$$

$$b = \frac{1}{27}\left[-I_1^3 + 9I_1I_2 - 27I_3\right]$$

and

$$\phi = \mathrm{Cos}^{-1}\left[-\mathrm{Sgn}(b)\sqrt{-\frac{27b^2}{4a^3}}\right]$$

Then

$$\lambda_k = \frac{I_1}{3} + 2\sqrt{-\frac{a}{3}}\left\{\mathrm{Cos}\left[\frac{\phi}{3} + (k-1)(120^\circ)\right]\right\} \quad \text{for } k = 1, 2, 3$$

This cubic equation solution process is used when the roots are real or, equivalently, whene

$$\frac{b^2}{4} + \frac{a^3}{27} < 0$$

The inequality expressed by Equation (90) can only be true if

$$a < -3\sqrt[3]{\frac{b^2}{4}} < 0$$

And so ϕ and λ_k expressed by Equations (88) and (89) are real.

2.4 DETERMINATION OF TAIT-BRYAN ANGLES WHICH WILL DIAGONALISE A REAL SYMMETRIC RANK TWO TENSOR

The problem addressed in this section is to determine the Tait-Bryan angles, corre to a specified Tait-Bryan rotation sequence, which will diagonalise a real symmetric (3x3) r two tensor. Specifically, if given a real symmetric (3x3) rank two tensor $\mathcal{D}$ expressed in co- system 1 such that

$$\mathcal{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} \quad (\text{(}$$

and a transformation matrix A^{21} from co-ordinate system 1 to co-ordinate system 2 so that

$$D = [A^{21}][\mathcal{D}][A^{21}]^T \quad (\text{(}$$

where D is $\mathcal{D}$ expressed in co-ordinate system 2.

then determine the Tait-Bryan angles corresponding to a specified Tait-Bryan rotation sequenc such that D is diagonal.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

As an example, consider the 1-2-3 Tait-Bryan rotation sequence for this development. The corre
${}^{321}A$ Tait-Bryan transformation matrix is defined by Equation (61). Then

$$A^{21} = {}^{321}A = {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)$$

and

$$D = {}^{321}A\,\mathcal{D}\,{}^{321}A^T = [{}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)]\,\mathcal{D}\,[{}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)]^T$$

where ${}^{321}A$ is the Tait-Bryan transformation which Diagonalises $\mathcal{D}$ and α, β, and γ are the ass
Tait-Bryan angles to be determined by this process.

Noting that

$$[A^N A^{N-1} \cdots A^3 A^2 A^1]^T = [A^1]^T [A^2]^T [A^3]^T \cdots [A^{N-1}]^T [A^N]$$

Equation (96) can be written

$$D = {}^{321}A\,\mathcal{D}\,{}^{321}A^T = {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)\,\mathcal{D}\,{}^{1}A^T(\alpha)\,{}^{2}A^T(\beta)\,{}^{3}A^T(\gamma)$$

The solution plan is to manipulate Equation (98) and search for elements which yield a tractable sy[stem] of simultaneous scalar equations from which α, β, and γ can be expressed as explicit scalar fu[nctions] of d_{11}, d_{22}, d_{33}, d_{12}, d_{13}, d_{23}, λ_1, λ_2, and λ_3. After considerable experimentation in the manipulatio[n of] this equation, the following procedure was selected.

Postmultiplication of Equation (98) by ${}^{321}A$ yields

$$D\,{}^{321}A = {}^{321}A\,\mathcal{D} \qquad (9$$

or equivalently

$$D\,{}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha) = {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)\,\mathcal{D} \qquad (1$$

where the relevant elements of Equation (100) (i.e., the scalar equations) are

Element of Equation (100)	Corresponding Scalar Equation
(1, 1)	$\lambda_1 \cos\beta \cos\gamma = d_{11}\cos\beta\cos\gamma + d_{12}\sin\alpha\sin\beta\cos\gamma + d_{12}\cos\alpha\sin\gamma - d_{13}\cos\alpha\sin\beta\cos\gamma + d_{13}\sin\alpha\sin\gamma$
(3, 1)	$\lambda_3 \sin\beta = d_{11}\sin\beta - d_{12}\sin\alpha\cos\beta + d_{13}\cos\alpha\cos\beta$
(3, 2)	$-\lambda_3 \sin\alpha\cos\beta = d_{12}\sin\beta - d_{22}\sin\alpha\cos\beta + d_{23}\cos\alpha\cos\beta$

Equation (102) and (103) can be rearranged to yield

$$(\lambda_3 - d_{11}) \operatorname{Sin}\beta = (d_{13} \operatorname{Cos}\alpha - d_{12} \operatorname{Sin}\alpha) \operatorname{Cos}\beta \qquad (1\ldots$$

and

$$d_{12} \operatorname{Sin}\beta = [(d_{22} - \lambda_3) \operatorname{Sin}\alpha - d_{23} \operatorname{Cos}\alpha] \operatorname{Cos}\beta \qquad (1\ldots$$

respectively.

Dividing Equation (105) by Equation (104) yields

$$\frac{d_{12}}{d_{11} - \lambda_3} = \frac{(d_{22} - \lambda_3) \operatorname{Sin}\alpha - d_{23} \operatorname{Cos}\alpha}{d_{12} \operatorname{Sin}\alpha - d_{13} \operatorname{Cos}\alpha} \qquad (1\ldots$$

From which

$$\boxed{\operatorname{Tan}\alpha = \frac{d_{12} d_{13} - d_{23}(d_{11} - \lambda_3)}{d_{12}^2 - (d_{22} - \lambda_3)(d_{11} - \lambda_3)}} \qquad (1\ldots$$

Equation (107) has two principal solutions, α_1 and α_2, of interest. If α_1 is the solution which
the first or second quadrant; that is if

$$\alpha_1 \in [0^\circ, 180^\circ) \qquad (1\ldots$$

Then

$$\lambda_2 = \lambda_1 + 180^\circ \in [180^\circ, 360^\circ) \qquad ($$

Equation (104) can be solved for Tan β; specifically

$$\boxed{\operatorname{Tan}\beta = \frac{d_{13}\operatorname{Cos}\lambda - d_{12}\operatorname{Sin}\lambda}{\lambda_3 - d_{11}}}$$

Assuming either λ_1 or λ_2, Equation (110) has two principal solutions, β_1 and β_2, of interes
β_1 is the solution which lies in the first or second quadrant; that is if

$$\beta_1 \in [0^\circ, 180^\circ)$$

Then

$$\beta_2 = \beta_1 + 180^\circ \in [180^\circ, 360^\circ)$$

Finally, an expression for Tan γ can be obtained from Equation (101) [i.e., $D^{321}A(1,1) = {}^{321}A\,D$
Specifically

$$\boxed{\operatorname{Tan}\gamma = \frac{(\lambda_1 - d_{11})\operatorname{Cos}\beta + (d_{13}\operatorname{Cos}\alpha - d_{12}\operatorname{Sin}\alpha)\operatorname{Sin}\beta}{d_{12}\operatorname{Cos}\alpha + d_{13}\operatorname{Sin}\alpha}} \qquad (11$$

Assuming specific solution values of α and β (i.e., α_1 or α_2 and β_1 or β_2), Equation (113) ha principal solutions, γ_1 and γ_2, of interest. If γ_1 is the solution which lies in the first or sec quadrant; that is if

$$\gamma_1 \in [0°, 180°) \qquad (11$$

Then

$$\gamma_2 = \gamma_1 + 180° \in [180°, 360°) \qquad (115$$

Up to this point, it has been assumed that the eigenvalues λ_1, λ_2, and λ_3 were all distinct. Th of repeated eigenvalues will now be examined. It can be shown, for example, that if $\lambda_1 = \lambda_2$ ((113) reduces to the indeterminate $0/0$ form not resolvable with L'Hospital's Rule. This m the rotation angle γ associated with λ_1 and λ_2 is indeterminate or not consequential and su that the tensor characteristic under consideration has an axial symmetry about the third axis. A

of the effect of repeated eigenvalues is facilitated using the following equation [Equation (1
which is easily derived from Equation (96) or (98).

$$\mathcal{D} = {}^{321}A^T D \, {}^{321}A = {}^{1}A^T(\alpha) \, {}^{2}A^T(\beta) \, {}^{3}A^T(\gamma) \, D \, {}^{3}A(\gamma) \, {}^{2}A(\beta) \, {}^{1}A(\alpha) \quad (1$$

For convenience, the elements of Equation (116) (i.e., the elements of $\mathcal{D}$) are listed below

$$d_{11} = \lambda_1 \cos^2\beta \cos^2\gamma + \lambda_2 \cos^2\beta \sin^2\gamma + \lambda_3 \sin^2\beta \quad (1$$

$$\left. \begin{aligned} d_{12} = d_{21} = {} & \lambda_1 \sin\alpha \sin\beta \cos\beta \cos^2\gamma + \lambda_1 \cos\alpha \cos\beta \sin\gamma \cos\gamma + \lambda_2 \sin\alpha \sin\beta \cos\beta \sin^2\gamma \\ & - \lambda_2 \cos\alpha \cos\beta \sin\gamma \cos\gamma - \lambda_3 \sin\alpha \sin\beta \cos\beta \end{aligned} \right\} \quad (1$$

$$\left. \begin{aligned} d_{13} = d_{31} = {} & -\lambda_1 \cos\alpha \sin\beta \cos\beta \cos^2\gamma + \lambda_1 \sin\alpha \cos\beta \sin\gamma \cos\gamma - \lambda_2 \cos\alpha \sin\beta \cos\beta \sin^2\gamma \\ & - \lambda_2 \sin\alpha \cos\beta \sin\gamma \cos\gamma + \lambda_3 \cos\alpha \sin\beta \cos\beta \end{aligned} \right\} \quad ($$

$$\left. \begin{aligned} d_{22} = {} & \lambda_1 \sin^2\alpha \sin^2\beta \cos^2\gamma + 2\lambda_1 \sin\alpha \cos\alpha \sin\beta \sin\gamma \cos\gamma + \lambda_1 \cos^2\alpha \sin^2\gamma \\ & + \lambda_2 \sin^2\alpha \sin^2\beta \sin^2\gamma - 2\lambda_2 \sin\alpha \cos\alpha \sin\beta \sin\gamma \cos\gamma + \lambda_2 \cos^2\alpha \cos^2\gamma \\ & + \lambda_3 \sin^2\alpha \cos^2\beta \end{aligned} \right\} \quad ($$

$$d_{23} = d_{32} = -\lambda_1 \sin\alpha\cos\alpha\sin^2\beta\cos^2\gamma + \lambda_1 \sin^2\alpha\sin\beta\sin\gamma\cos\gamma - \lambda_1\cos^2\alpha\sin\beta\sin\gamma\cos\gamma$$
$$+ \lambda_1\sin\alpha\cos\alpha\sin^2\gamma - \lambda_2\sin\alpha\cos\alpha\sin^2\beta\sin^2\gamma - \lambda_2\sin^2\alpha\sin\beta\sin\gamma\cos\gamma$$
$$+ \lambda_2\cos^2\alpha\sin\beta\sin\gamma\cos\gamma + \lambda_2\sin\alpha\cos\alpha\cos^2\gamma - \lambda_3\sin\alpha\cos\alpha\cos^2\beta$$
(1

$$d_{33} = \lambda_1\cos^2\alpha\sin^2\beta\cos^2\gamma - 2\lambda_1\sin\alpha\cos\alpha\sin\beta\sin\gamma\cos\gamma + \lambda_1\sin^2\alpha\sin^2\gamma$$
$$+ \lambda_2\cos^2\alpha\sin^2\beta\sin^2\gamma + 2\lambda_2\sin\alpha\cos\alpha\sin\beta\sin\gamma\cos\gamma + \lambda_2\sin^2\alpha\cos^2\gamma$$
$$+ \lambda_3\cos^2\alpha\cos^2\beta$$
(12

First, consider the case where all the eigenvalues are equal, if indeed this case makes sense o possible. Substitution of $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ into Equations (117) through (122) yields the

$$\mathcal{D} = D \qquad \text{or equivalently} \qquad \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{for } \lambda_1 = \lambda_2 = \lambda_3$$

and

$$^{321}A \equiv E_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{for } \lambda_1 = \lambda_2 = \lambda_3$$

This means that all three rotation angles, α, β, and γ, drop out of the transformation and so values, which are indeterminate, are not consequential. This suggests that the tensor charac under consideration have symmetry about a point, and that their representation is the same regar the co-ordinate system.

Next, consider the case where $\lambda_1 = \lambda_2 \neq \lambda_3$. It was already noted that in this case Equation became indeterminate. This result is arrived at by first substituting $\lambda = \lambda_1 = \lambda_2 \neq \lambda_3$ into E (117) through (122). The resulting elements of $\mathcal{D}$ become

$$d_{11} = \lambda \cos^2\beta + \lambda_3 \sin^2\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

$$d_{12} = d_{21} = (\lambda - \lambda_3) \sin\alpha \sin\beta \cos\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

$$d_{13} = d_{31} = -(\lambda - \lambda_3) \cos\alpha \sin\beta \cos\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

$$d_{22} = \lambda(\sin^2\alpha \sin^2\beta + \cos^2\alpha) + \lambda_3 \sin^2\alpha \cos^2\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

$$d_{23} = d_{32} = (\lambda - \lambda_3) \sin\alpha \cos\alpha \cos^2\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

$$d_{33} = \lambda(\cos^2\alpha \sin^2\beta + \sin^2\alpha) + \lambda_3 \sin^2\alpha \cos^2\beta \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

From Equations (125) through (130), it can be seen that the third rotation angle, γ, comple
disappears from the transformation when $\lambda_1 = \lambda_2 \neq \lambda_3$. This means that the transformation ex
by Equation (116) is not dependent on γ in this case.

Substitution of Equations (125), (126), and (127) into Equation (113) leads to the previous
stated conclusion that $\operatorname{Tan}\gamma$, as expressed by Equation (113), is of the $0/0$ indeterminate for
$\lambda_1 = \lambda_2 \neq \lambda_3$.

Simplified expressions for α and β can be obtained from Equations (125), (126), and (127).
Equation (126) divided by Equation (127) yields

$$\boxed{\operatorname{Tan}\alpha = -\frac{\dot{d}_{21}}{\dot{d}_{31}}} \qquad \underline{\text{for } \lambda_1 = \lambda_2 \neq \lambda_3}$$

Equation (131) has two principal solutions, α_1 and α_2, of interest. If α_1 is the solution which
the first or second quadrant; that is if

$$\alpha_1 \in [0°, 180°)$$

Then

$$\alpha_2 = \alpha_1 + 180° \in [180°, 360°)$$

Rearrangement of Equation (125) with appropriate trigonometric substitution yields

$$\cos 2\beta = \frac{2d_{11} - \lambda - \lambda_3}{\lambda - \lambda_3} \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

If $\alpha \neq n(180°)$ for $n \in \{\text{integers}\}$, or equivalently if $\sin\alpha \neq 0$, Equation (126) ca[n be]
written as

$$\sin 2\beta = \frac{2d_{12}}{(\lambda - \lambda_3)\sin\alpha} \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

If $\alpha \neq (2n+1)(180°)$ for $n \in \{\text{integers}\}$, or equivalently if $\cos\alpha \neq 0$, Equation (127) c[an be]
written as

$$\sin 2\beta = \frac{-2d_{13}}{(\lambda - \lambda_3)\cos\alpha} \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

With the appropriate selection of Equation (135) or (136), β is expressed

$$\boxed{\beta = \frac{1}{2}\,\mathrm{Arc}\left(\sin 2\beta,\ \cos 2\beta\right)} \qquad \text{for } \lambda_1 = \lambda_2 \neq \lambda_3$$

Equation (137) has two principal solutions, β_1 and β_2, of interest. If β_1 is the solution whic in the first or second quadrant; that is if

$$\beta_1 \in [0^\circ, 180^\circ)$$

then

$$\beta_2 = \beta_1 + 180^\circ \in [180^\circ, 360^\circ)$$

An approach similar to that employed to investigate the case where $\lambda_1 = \lambda_2 \neq \lambda_3$ was employed t investigate the effect of repeated eigenvalues for the cases $\lambda_2 = \lambda_3 \neq \lambda_1$ and $\lambda_3 = \lambda_1 \neq \lambda_2$. Althoug was a slight simplification to Equations (117) through (122), no significant simplification of the exhibited by Equations (125) through (130) for $\lambda_1 = \lambda_2 \neq \lambda_3$, where a rotation angle was complet dropped from the transformation, was discovered. Consequently α, β, and γ can be determined by procedure described by Equations (107) through (115) for cases where $\lambda_2 = \lambda_3 \neq \lambda_1$ or $\lambda_3 = \lambda_1$

3.0 RESULTS

A process to define the Tait-Bryan rotation angles which diagonalise a real symmetric rank two tensor $\mathcal{D}$ was identified and is described herein. The first step in this process is determine the eigenvalues of $\mathcal{D}$. This is accomplished with the process described by Equations through (81) and Equations (86) through (89).

For a 1-2-3 Tait-Bryan rotation sequence such as that defined by Equation (95) (i.e., ${}^{321}A = {}^{3}A(\gamma)\,{}^{2}A(\beta)\,{}^{1}A(\alpha)$), the Tait-Bryan angles which diagonalise $\mathcal{D}$ can be defined according to

1. If the eigenvalues are all distinct (i.e., if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$), or if $\lambda_1 \neq \lambda_2$, α, β, a can be determined by the procedure described by Equations (107) through (115).
2. If $\lambda_1 = \lambda_2 \neq \lambda_3$, α and β can be determined by the procedure described by Equ (131) through (139) and γ is indeterminate.
3. If $\lambda_1 = \lambda_2 = \lambda_3$, α, β, and γ are all indeterminate.

For the other Tait-Bryan rotation sequences (i.e., 2-3-1, 3-1-2, 3-2-1, 1-3-2, and 2-1-3 procedures similar to that used for the 1-2-3 sequence exist and can readily be defined.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
April 1996

3. REPORT TYPE AND DATES COVERED
Technical Memorandum

4. TITLE AND SUBTITLE
Tait-Bryan Angles Required for Diagonalisation of a Real Symmetric (3 × 3) Rank Two Tensor

5. FUNDING NUMBERS
505-59-36

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
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8. PERFORMING ORGANIZATION REPORT NUMBER
A-961266

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONIT AGENCY REPORT NU
NASA TM-1103

11. SUPPLEMENTARY NOTES
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12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified — Unlimited
Subject Category 37

12b. DISTRIBUTION CO

13. ABSTRACT *(Maximum 200 words)*

The process to define the Tait-Bryan angles required to diagonalise a real symmetric (3×3) rank two tensor was identified. This process is applicable to commonly encountered ten such as inertia tensors, rank two (3 × 3) covariance tensors, and quadratic coefficient tensors. Tait-Bryan angles, which are any permutation of 1-2-3 rotation angles, are used to de the pitch, flap, and lead-lag angles in helicopter rotor systems. The Tait-Bryan angles differ from the more familiar Euler angles which are any permutation of 3-1-3 rotation angl

14. SUBJECT TERMS
Tait-Bryan angles, Rotation angles, Transformation matrix reductions, Rotation angles from a transformation matrix, Tait-Bryan angles from a transformation matrix

15. NUMBER OF PA
52

16. PRICE CODE
A04

17. SECURITY CLASSIFICATION OF REPORT
Unclassified

18. SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

19. SECURITY CLASSIFICATION OF ABSTRACT

20. LIMITATION OF

NSN 7540-01-280-5500

Standard Form 2
Prescribed by ANSI Std
298-102